THE THREE DIVERGENCE FREE MATRIX FIELDS PROBLEM

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Abstract. We prove that for any connected open set $\Omega \subset \mathbb{R}^n$ and for any set of matrices $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $m \geq n$ and $\mathrm{rank}(A_i - A_j) = n$ for $i \neq j$, there is no non-constant solution $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$, called exact solution, to the problem

$$\mathrm{Div}B = 0 \quad \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m) \quad \text{and} \quad B(x) \in K \text{ a.e. in } \Omega\,.$$

In contrast, A. Garroni and V. Nesi [10] exhibited an example of set K for which the above problem admits the so-called approximate solutions. We give further examples of this type.

We also prove non-existence of exact solutions when K is an arbitrary set of matrices satisfying a certain algebraic condition which is weaker than simultaneous diagonalizability.

Key words: Differential inclusions, Phase transitions, Homogenization.

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Contents

1. Introduction

The problem of characterizing solenoidal matrix fields which take values in a finite set of matrices, has been recently considered by A. Garroni and V. Nesi. This kind of problem is analogous to that on curl free matrix fields in which one asks whether a Lipschitz mapping using a finite number of gradients exists. Here the differential constraint of being the gradient of a mapping, and hence a curl free matrix field, is replaced by that of being a divergence free matrix field (i.e. a matrix valued function whose rows are divergence free in the distributional sense). To describe the problem we begin with some definitions.

Definition 1. Given two integers $m, n \geq 2$, a set of real $m \times n$ matrices $K \subset \mathbb{M}^{m\times n}$ and a bounded open set Ω in $\mathbb{R}^n$, we say that any $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ satisfying

$$
\begin{cases} \mathrm{Div} B = 0 & \text{in } \mathcal{D}'(\Omega, \mathbb{R}^m)\,, \\ B(x) \in K & \text{a.e. in } \Omega\,, \\ B \text{ is non-costant}\,, \end{cases} \tag{1.1}
$$

is an exact solution of (1.1). We say that K is rigid for exact solutions if there is no solution to (1.1).

Definition 2. We say that Problem (1.1) admits an approximate solution if there exists a uniformly bounded sequence $\{B_h\} \subset L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that

$$
\begin{cases} \mathrm{Div} B_h \to 0 & \text{in } W^{-1,\infty}(\Omega, \mathbb{R}^m), \\ \mathrm{dist}(B_h, K) \to 0 & \text{in measure}, \\ \mathrm{dist}(B_h, A) \not\to 0 & \text{in measure, for every } A \in K. \end{cases} \tag{1.2}
$$

We say that K is rigid for approximate solutions of (1.1) if there is no solution to (1.2).

We remark that if K is rigid for approximate solutions and there exists a sequence $\{B_h\}$ satisfying the first two conditions of (1.2), then any accumulation point of the sequence $\{B_h\}$ has to be a constant matrix in K.

Let us briefly describe the situation in the context of the "gradient problem", that is: find $f \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ such that $Df \in K$ a.e. in Ω and f is not affine. It is well-known in this setting, that a sufficient condition for a set $K = \{A_1, \ldots, A_N\} \in \mathbb{M}^{m\times n}$, $N \leq 3$, to be rigid is that $\mathrm{rank}(A_i - A_j) > 1$ for $i \neq j$. The condition $\mathrm{rank}(A_i - A_j) = 1$ is called rank-1 connectedness. J. M. Ball and R. D. James studied in detail the case $N = 2$ and proved, under the latter assumption, a rigidity result both for exact and for approximate solutions (see [4]). For $N = 3$ rigidity still holds. The following theorem is due to Šverák and will be used later.

Theorem 1.1. (V. Šverák, [14]). *Let $\Omega \subset \mathbb{R}^n$ be an open connected set and let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $\mathrm{rank}(A_i - A_j) > 1$ for $i \neq j$. If $f \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ satisfies $Df \in K$ a.e., then Df is constant.*
Let $p > 2$ and let $f_h \rightharpoonup f$ in $W^{1,p}(\Omega, \mathbb{R}^m)$. If $\mathrm{dist}(Df_h, K) \to 0$ in measure, then $Df_h \to A_i$ in measure, for some $i \in \{1, 2, 3\}$.

The previous result, specialized to the case $m = n = 2$, will be a crucial tool in the proof of Theorem 1.7.

For completeness let us also recall that for $N = 4$ rigidity still holds for exact solutions (see [8]) but it can fail for approximate ones and a suitable choice of $\{A_1, A_2, A_3, A_4\}$

(see [16], [17]). The case $N = 5$ is nicely illustrated in [11] by a non-rigid five point configuration without any rank-1 connection.

Remark 1.2. All the previous works provide results also for the "divergence problem" when the working space is $\mathbb{M}^{m\times 2}$, since any set of solenoidal matrix fields defines a set of gradients via right-multiplication by $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The problems are no longer equivalent if $m \geq n > 2$. The right notion of connectedness which comes into play, in this case, is that of the so called rank-$(n-1)$ connectedness. Given $A_1, A_2 \in \mathbb{M}^{m\times n}$, with $m \geq n$ and $\operatorname{rank}(A_1 - A_2) \leq n - 1$, one can construct solenoidal matrix fields which take both the values A_1 and A_2 on a set of positive measure (indeed one can check that simple laminates work). In contrast, if $\operatorname{rank}(A_1 - A_2) = n$, the following rigidity result holds.

Proposition 1.3. (A. Garroni, V. Nesi, [10]). *Let Ω be an open and connected set in $\mathbb{R}^n$. Let $A_1, A_2 \in \mathbb{M}^{m\times n}$, with $m \geq n$ and $\operatorname{rank}(A_1 - A_2) = n$. Let $B : \Omega \to \{A_1, A_2\}$ be a measurable function satisfying $\operatorname{Div} B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$. Then B is constant.*

The case when K is made of two matrices has been completely solved and given a negative answer also for what concerns approximate solutions. The following proposition establishes rigidity for approximate solutions under the hypothesis of rank-$(n-1)$ disconnectedness.

Proposition 1.4. (A. Garroni, V. Nesi, [10]). *Let Ω be a bounded open and connected set in $\mathbb{R}^n$, and let $K = \{A_1, A_2\} \subset \mathbb{M}^{m\times n}$, $m \geq n \geq 1$, be such that $\operatorname{rank}(A_1 - A_2) = n$. Let B_h be a sequence weakly convergent to B in $L^p(\Omega, M^{m\times n})$, with $p > 1$, such that*

$$\operatorname{Div} B_h \to 0 \quad \textit{strongly in } W^{-1,p}(\Omega, \mathbb{R}^m)$$

and

$$\operatorname{dist}(B_h, K) \to 0 \quad \textit{in measure}.$$

Then

$$B_h \to A_1 \quad \textit{or} \quad B_h \to A_2 \quad \textit{in measure}.$$

So far everything seems to be parallel the "gradient problem", but the case when K consists of three matrices turns out to be different. Indeed one can construct a sequence of matrix fields which are divergence free and whose distance from the set K approaches zero. In other words, approximate solutions exist for a suitable choice of $\{A_1, A_2, A_3\}$. The following result clarifies the situation.

Lemma 1.5. (A. Garroni, V. Nesi, [10]). *Given $m \geq n \geq 3$, there exist three pair-wise rank-n connected $m \times n$ matrices A_1, A_2, A_3, and there exists a sequence $B_h \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ such that setting $K = \{A_1, A_2, A_3\}$, one has*

$$\operatorname{dist}(B_h, K) \to 0 \qquad \textit{strongly in } L^p(\Omega),\ \forall p \geq 1, \tag{1.3}$$

$$\operatorname{Div} B_h \to 0 \qquad \textit{strongly in } W^{-1,p}(\Omega; \mathbb{R}^m),\ \forall p \geq 1, \tag{1.4}$$

and $B_h \overset{*}{\rightharpoonup} B$ in L^∞, with $B \neq A_i$ for any $i = 1, 2, 3$.

Remark 1.6. (A. Garroni, V. Nesi, [10]). In Lemma 1.5 one can achieve the stronger requirement $\operatorname{Div} B_h = 0$ rather then (1.4), by suitably projecting the fields B_h onto Divergence-free matrix fields.

The explicit formula for A_1, A_2, A_3, can be found in the last section of this paper (see Remark 4.3).

Next, we state the main theorem of our paper, namely a rigidity result for three-valued matrix fields under the assumption of rank-$(n-1)$ disconnectedness.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{m\times n}$, with $m \geq n$ and $\operatorname{rank}(A_i - A_j) = n$ for $i \neq j$. If $B : \Omega \to K$ is a measurable function satisfying $\operatorname{Div} B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$, then B is constant.*

In this paper we will mainly deal with the problem of non-existence of exact solutions. In addition to the previous theorem, we prove a rigidity result for a particular class of matrix fields taking an arbitrary number of values. This is the precise statement.

Theorem 1.8. *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and let $K \subset \mathbb{M}^{m\times n}$ be bounded, with $m \geq n$ and rank$(A_i - A_j) = n$ for every $A_i, A_j \in K$, with $i \neq j$. Suppose that K satisfies the following condition:*
there exist $n-1$ independent hyperplanes $\pi_1, \ldots, \pi_{n-1}$ in $\mathbb{R}^n$, and $n-1$ vector subspaces $\tau_1, \ldots, \tau_{n-1}$ of dimension $n-1$ in $\mathbb{R}^m$, such that

$$A_i : \pi_r \to A_i(\pi_r) \subseteq \tau_r\,, \quad \textit{for every } A_i \in K \textit{ and } r = 1, \ldots n-1\,. \tag{1.5}$$

Then every measurable matrix field $B : \Omega \to K$ satisfying $\operatorname{Div} B = 0$ in $\mathcal{D}'(\Omega, \mathbb{R}^m)$ is constant.

We remark that Theorem 1.8, specialized to the case $m = n = 2$, reduces itself to a result which is well-known in the setting of the gradient problem (see [8], Lemma 5). An easy corollary of this theorem is that a set of simultaneously diagonalizable rank-n connected matrices is rigid for exact solutions.

The study of more general linear differential constraints on the matrix field B is just beginning. The interested reader is referred to [5] for results in this direction.

The plan of the present paper is as follows.

In Section 2, we present an algebraic argument which implies that the right dimension to study the problem is $n \times n$ (see Lemma 2.4). Next we remark that the condition of being divergence-free is invariant under any orthogonal change of variables (Remark 2.2). Using this invariance, in order to prove Theorem 1.7 it is enough to consider a very special situation. This kind of argument does not work for an arbitrary number of matrices and one does in fact expect that rigidity fails for a sufficiently large number of them. Yet, under the assumptions of Theorem 1.8, one can prove that rigidity still holds for an arbitrary number of values and actually even for a continuum of them. For the reader's convenience we give, in Lemma 2.1, the Gauss-Green formula for L^∞ fields, which will be the main ingredient in the proof of this result.

In Section 3, we give the proofs of Theorem 1.7 and Theorem 1.8.

The final section departs from the main focus of the paper. Indeed, in the spirit of Lemma 1.5, we address the problem of finding approximate solutions to the "three divergence problem". More precisely, we show that the construction used by Garroni and Nesi actually applies to a larger class of sets K. Theorem 1.9 gives a characterization of all such K's, which turn out to be non-rigid for approximate solutions.

Theorem 1.9. *For every q_1, q_2, $q_3 \in (0,1)$, let $A \in \mathbb{M}^{3\times 3}$ be defined as follows*

$$A = \frac{1}{q_3}\left[\left(1-\prod_{i=1}^{3}(1-q_i)\right) G^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3)I\right],$$

where $\lambda_1 = 0\,, \lambda_2 = 1/(1-q_1)\,, \lambda_3 = q_2/(q_1+q_2-q_1q_2)$, and G is an arbitrary matrix in $GL(3)$. Then, for every $M \in \mathbb{M}^{3\times 3}$ and $N \in GL(3)$, the set

$$K = \{M\,, N+M\,, NA+M\}$$

is non-rigid for approximate solutions.

2. Preliminaries

In this Section we set some notations and a few preliminary results needed in the proof of the main results of Section 3.

Throughout this paper Ω is an open connected subset of $\mathbb{R}^n$. We denote by $\mathbb{M}^{m\times n}$ the set of the real $m \times n$ matrices; 0 and I will indicate the zero matrix and the identity matrix in $\mathbb{M}^{n\times n}$ respectively. Left-multiplication of A times a vector $v \in \mathbb{R}^n$ is denoted by $A \cdot v$. The symbol $\langle v, w\rangle$ denotes the standard inner product in $\mathbb{R}^n$.

For every measurable subset E of $\mathbb{R}^n$, $|E|$ is the n-dimensional Lebesgue measure of E while, for $s \in \mathbb{R}^+$, $\mathcal{H}^s(E)$ is its s-dimensional Hausdorff measure.
Given a function $f \in L^1(\Omega, \mathbb{R})$, we say that $x \in \Omega$ is a Lebesgue point for f, and that $\lambda(x) \in \mathbb{R}$ is the Lebesgue value of f at x, if

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - \lambda(x)|\, dy = 0,$$

where $B_r(x)$ is the open ball of radius r and center x and the symbol $\fint_{B_r(x)}$ stands for $\frac{1}{|B_r(x)|}\int_{B_r(x)}$. This definition extends in the obvious way to vector valued functions.
It is well-known that the set of Lebesgue points for f, which from now on we will denote by $\mathbb{L}(f)$, has full measure in Ω. For every $k \in \mathbb{N}$ and $f \in L^1(\Omega, \mathbb{R}^k)$, we will denote by $\tilde{f}$ a Lebesgue representative of f (i.e. $\tilde{f}(x) = \lambda(x)$ for every $x \in \mathbb{L}(f)$), so that $\tilde{f}$ coincides with f a.e. in Ω. For more details we refer the reader to [9].

Recall that a vector field $f \in L^\infty(\Omega, \mathbb{R}^n)$ is said to be divergence free if for every $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega \langle f(x), \nabla\varphi(x)\rangle\, dx = 0.$$

For the reader's convenience, we prove a Gauss-Green formula in the particular setting of our problem (much more general results can be found in [1]). In the sequel the symbol $\nu(x)$ will denote the outward normal to a given surface at the point x.

Lemma 2.1. *Let $f \in L^\infty(\Omega, \mathbb{R}^n)$ be a divergence free vector field, and let $U \subset \overline{U} \subset \Omega$ be open with Lipschitz boundary. Suppose that $\mathcal{H}^{n-1}(\mathbb{L}(f) \cap \partial U) = \mathcal{H}^{n-1}(\partial U)$. Then*

$$\int_{\partial U} \langle \tilde{f}(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0\,.$$

Proof. Consider a sequence $\{\rho_n\}$ of mollifiers and set $f_n := f * \rho_n$. We have

$$\mathrm{div} f_n = \mathrm{div} f * \rho_n = 0.$$

The standard Gauss-Green formula for smooth functions on a Lipschitz domain yields

$$\int_{\partial U} \langle f_n(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0. \tag{2.1}$$

It is easy to see that $f_n(x) \to \tilde{f}(x)$ for all $x \in \partial U \cap \mathbb{L}(f)$. Passing to the limit in (2.1) and using the dominated convergence Theorem, we get

$$\int_{\partial U} \langle \tilde{f}(s), \nu(s)\rangle\, d\mathcal{H}^{n-1}(s) = 0.$$

□

Remark 2.2. Let $B \in L^\infty(\Omega, \mathbb{M}^{m\times n})$ be a divergence free matrix field and let R be an orthogonal matrix in $\mathbb{M}^{n\times n}$. Using a convolution argument as in Lemma 2.1 and the

classical chain rule formula for smooth functions, one can check that for every $C \in \mathbb{M}^{m\times n}$ and $F \in \mathbb{M}^{n\times m}$, the matrix field $\widehat{B} : \{y \in \mathbb{R}^n \,|\, y = R^T x,\, x \in \Omega\} \to \mathbb{M}^{n\times n}$ defined by

$$\widehat{B}(y) := R^T F\big(B(Ry) + C\big)R \tag{2.2}$$

is divergence free.

The next lemma shows that, given a set $K \subset \mathbb{M}^{m\times n}$, the property of rank-$(n-1)$ disconnectedness is preserved under left multiplication by suitable matrices in $\mathbb{M}^{n\times m}$.

Lemma 2.3. *Let $K \subset \mathbb{M}^{m\times n}$ be at most countable, with $m > n$ and $\operatorname{rank}(A_i - A_j) = n$ for every A_i, $A_j \in K$, with $i \neq j$. Then there exists $F \in \mathbb{M}^{n\times m}$ such that $\operatorname{rank}(FA_i - FA_j) = n$ for every A_i, $A_j \in K$ with $i \neq j$.*

Proof. Since the image of $A_i - A_j$ is a n-dimensional subspace of $\mathbb{R}^m$, we can always find a linear operator $F : \mathbb{R}^m \to \mathbb{R}^n$ such that $\operatorname{Ker}(F(A_i - A_j)) = \operatorname{Ker}(F) \cap \operatorname{Im}(A_i - A_j) = \{0\}$ for every $i \neq j$. Then $\operatorname{rank}(FA_i - FA_j) = n$ for every A_i, $A_j \in K$ with $i \neq j$. □

The previous results show that, as long as we consider discrete valued matrix fields, we can always set the problem in the space of square matrices. This is the claim of the next Lemma.

Lemma 2.4. *Let $K \subset \mathbb{M}^{m\times n}$ be at most countable, with $m > n$ and $\operatorname{rank}(A_i - A_j) = n$ for every $A_i, A_j \in K$, with $i \neq j$. Suppose we are given a divergence free matrix field $B : \Omega \to K$. Then there exist a divergence free matrix field $B' : \Omega \to K'$, where $K' \subset \mathbb{M}^{n\times n}$, $\operatorname{card}(K') = \operatorname{card}(K)$, and $\operatorname{rank}(A'_i - A'_j) = n$ for every $A'_i, A'_j \in K'$, with $i \neq j$.*

Proof. Apply Lemma 2.3 and use Remark 2.2 with $R = I$ and $C = 0$. □

3. Proofs of the main results

In this section we give the proofs of Theorem 1.7 and Theorem 1.8.

Proof of Theorem 1.7. By Lemma (2.4) it is enough to consider the case $m = n > 2$. Moreover, due to the local character of our problem, we can make any convenient change of variables. Hence, as it is customary in this kind of problems, we begin with some reductions to special cases. We use Remark 2.2 choosing $F = (A_2 - A_1)^{-1}$ and $C = -A_1$. In this way we can assume that $A_1 = 0$ and $A_2 = I$. Moreover, since for any linear operator in $\mathbb{R}^n$ there exists a two-dimensional invariant subspace, we can choose the orthogonal matrix R in (2.2) so that A_3 is of the form

$$A_3 := A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}. \tag{3.1}$$

and $\mathrm{rank}(A) = \mathrm{rank}(A - I) = n$. By assumptions we can write

$$B = \chi_{E_1} 0 + \chi_{E_2} I + \chi_{E_3} A \,,$$

where E_i are disjoint measurable sets and $E_1 \cup E_2 \cup E_3 = \Omega$. Next remark that, due to (3.1), the first two equations of $\mathrm{Div}B = 0$ only involve derivatives with respect to directions e_1 and e_2. Roughly speaking, the idea is to use these informations to conclude that B does not depend on the variables x_1 and x_2. This allows us to make a "projection" of the original problem into a lower dimensional space (actually a section of Ω) and to proceed by induction on the dimension n, using, at the final step, that by Theorem 1.1 and Remark 1.2, rigidity for exact solutions holds for $n = 2$.

Let us proceed with the formal proof. Set $n > 3$ and suppose that rigidity holds in $\mathbb{M}^{(n-1)\times(n-1)}$. We want to prove that B is constant. For every $x \in \Omega$, let Q be a open coordinate cube centered in x and such that $\overline{Q} \subset \Omega$. Without loss of generality we assume that $x = 0$, so that $Q = (-l, l)^n$ for some positive l.

Let $\{\rho_i\}$ be a sequence of mollifiers and set $B_i := \rho_i * B$. For i large enough we have that B_i is well defined in Q, $B_i \in C^\infty(Q, \mathbb{M}^{n\times n})$ and $\mathrm{Div}B_i = 0$ in Q in the classical sense. Then fix $\overline{x} := (\overline{x}_3, \cdots, \overline{x}_n) \in (-l, l)^{n-2}$ such that for $\mathcal{H}^2$-a.e. $(x_1, x_2) \in (-l, l)^2$, $(x_1, x_2, \overline{x}) \in \mathbb{L}(B)$. By Fubini's Theorem, this is possible for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in (-l, l)^{n-2}$, since, as already remarked, $\mathbb{L}(B)$ has full measure in Ω. Now consider the field $\underline{B} : (-l, l)^2 \to \mathbb{M}^{2\times 2}$ defined by

$$\underline{B}(x_1, x_2) := \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix} (x_1, x_2, \overline{x}), \tag{3.2}$$

where $\tilde{b}_{kl}$ is a Lebesgue representative of the kl-entry of the matrix B. Then let b^i_{kl} be the kl-entry of B_i and set

$$\underline{B_i}(x_1, x_2) := \begin{pmatrix} b^i_{11} & b^i_{12} \\ b^i_{21} & b^i_{22} \end{pmatrix} (x_1, x_2, \overline{x}). \tag{3.3}$$

Since $\underline{B_i} \in C^\infty((-l, l)^2, \mathbb{M}^{2\times 2})$ and $\mathrm{Div}\ \underline{B_i} = 0$ in $(-l, l)^2$, we have

$$\int_{(-l,l)^2} \underline{B_i} \cdot \nabla\varphi \, dx_1 dx_2 = 0 \qquad \forall \varphi \in C^\infty_0(-l, l)^2. \tag{3.4}$$

Moreover $\underline{B_i}(x_1, x_2)$ converges to $\underline{B}(x_1, x_2)$ at every $(x_1, x_2) \in (-l, l)^2$ such that $(x_1, x_2, \overline{x}) \in \mathbb{L}(B)$. Then passing to the limit for $i \to \infty$ in (3.4) and using the dominated convergence Theorem we get

$$\int_{(-l,l)^2} \underline{B}(x_1, x_2) \cdot \nabla\varphi(x_1, x_2) \, dx_1 dx_2 = 0 \qquad \forall \varphi \in C^\infty_0(-l, l)^2. \tag{3.5}$$

By Theorem 1.1 we have that $\underline{B}$ is constant and hence $\widetilde{B}$ is constant on the section $(x_1, x_2, \overline{x})$. Since this is true for $\mathcal{H}^{n-2}$-a.e. $\overline{x} \in (-l,l)^{n-2}$, we deduce that $\widetilde{B}$ does not depend on (x_1, x_2) in Q. In particular we have

$$\frac{\partial \chi_{E_i}}{\partial x_1} = 0 \qquad \text{in } \mathcal{D}'(Q). \tag{3.6}$$

From (3.6), it is easy to see that there exist three measurable sets E_1', E_2', E_3' in $(-l,l)^{n-1}$ such that

$$E_i = (-l,l) \times E_i' \quad \text{ a.e.} \tag{3.7}$$

Now call $0'$, I', A' the $n \times (n-1)$-minors of the matrices 0, I, A respectively, obtained by eliminating the first column of each matrix. Notice that $\mathrm{rank}(A') = \mathrm{rank}(A' - I') = n-1$. Then set

$$B' := 0'\chi_{E_1'} + I'\chi_{E_2'} + A'\chi_{E_3'}\,.$$

Let us emphasize that B' leaves in a space of dimension $n-1$. Combining (3.6) with the equation $\mathrm{Div}B = 0$ in $\mathcal{D}'(Q, \mathbb{R}^n)$, one concludes that B' satisfies

$$\mathrm{Div}B' = 0 \quad \text{ in } \mathcal{D}'((-l,l)^{n-1}, \mathbb{R}^n)\,. \tag{3.8}$$

By Lemma 2.3, there exists $F \in M^{(n-1)\times n}$ such that

$$\mathrm{rank}(FI') = \mathrm{rank}(FA' - FI') = n-1\,. \tag{3.9}$$

Then set

$$\begin{aligned} A_2' &:= FI'\,, \\ A_3' &:= FA'\,, \\ B_{(n-1)} &:= 0\chi_{E_1'} + A_2'\chi_{E_2'} + A_3'\chi_{E_3'}\,. \end{aligned}$$

By (3.8) and (3.9), it follows that $B_{(n-1)}$ is an exact solution of the problem

$$\mathrm{Div}B_{(n-1)} = 0 \quad \text{ in } \mathcal{D}'((-l,l)^{n-1}, \mathbb{R}^{n-1})\,, \quad B_{(n-1)} \in \{0, A_2', A_3'\} \subset \mathbb{M}^{(n-1)\times(n-1)}\,. \tag{3.10}$$

By the inductive assumption, problem (3.10) is rigid. Then $|(-l,l)^{n-1}| = |E_i'|$ for some i and hence, by (3.7), $|Q| = |E_i|$. By the arbitrariness of Q we conclude that B is constant. $\square$

Remark 3.1. Before giving the proof of Theorem 1.8, we make some considerations about its assumption (1.5). We want to show that, even in this case, we can reduce to a very special situation. Indeed, condition (1.5) implies that $\mathrm{Im}(A_i) = \mathrm{Im}(A_j)$ for every $A_i, A_j \in K$. We can thus apply the same argument as in Lemma 2.4, although no requirement is made on the cardinality of the set K. More precisely, we fix any two of the matrices in K, say A_1 and A_2, and choose a matrix $F \in \mathbb{M}^{n\times m}$ such that $F(A_1 - A_2) = I$.

For such an F, we have that $FA_i \in \mathbb{M}^{n\times n}$ and $\mathrm{rank}(FA_i - FA_j) = n$ for every $A_i, A_j \in K$. Moreover, the hyperplanes $\pi_1, \dots, \pi_{n-1}$ are preserved under the action of every FA_i.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8. By Remark 2.2 and Remark 3.1, we can assume that $K \subset \mathbb{M}^{n\times n}$ and $A_i(\pi_r) \subseteq \pi_r$ for every $A_i \in K$ and $r = 1, \dots, n-1$. For every r, let v_r be the unit vector orthogonal to π_r. We want to prove that B does not depend on any of the directions v_r, which are independent by assumptions. Then B would only depend on one direction, but the condition of being divergence free will imply that B is constant. We will only check the statement for one vector v_r. Choose any of the vectors v_r and a real number q, and by contradiction assume that there exist two points P, $Q \in \mathbb{L}(B)$, with $P - Q = qv_r$ and such that $\tilde{B}(P) \neq \tilde{B}(Q)$; for instance suppose that $\tilde{B}(P) = A_1$ and $\tilde{B}(Q) = A_2$.

Let us briefly digress to explain the idea of the proof in an informal way. We want to apply the Gauss-Green formula in a small cylinder with axis parallel to v_r and bases centered in P and Q respectively. Computing the flux of B through the boundary of the cylinder we will check that its contribution through the bases can never compensate the contribution through the lateral boundary, the former being a vector parallel to $(A_1 - A_2)v_r$, the latter belonging to the hyperplane π_r. This idea, however, will require some technical efforts, as B may be non-constant on the bases. What we do, in fact, is to consider a sequence of cylinders with vanishing radii. The contradiction will arise for a sufficiently small radius.

Now we continue the formal proof. To simplify the notations we will assume that $P = (0, \dots, 0)$ and $v_r = e_n$, so that $Q = (0, \dots, 0, q)$.
Let $\rho(x) := (x_1^2 + \cdots + x_{n-1}^2)^{1/2}$ and, for every $r \in \mathbb{R}^+$, set

$$C(r) := \{x \in \mathbb{R}^n : \quad 0 \leq \rho(x) \leq r,\ 0 \leq x_n \leq q\}.$$

Since $\widetilde{B}(P) = A_1$ and $\widetilde{B}(Q) = A_2$, we can find a cylinder on whose bases the mean value of B is A_1' and A_2' respectively, with $|A_1 - A_1'|$ and $|A_2 - A_2'|$ arbitrarily small (this can be checked by using Fubini's theorem). More precisely, for every given $\delta > 0$, we can find a radius $r_\delta \in \mathbb{R}^+$ and a vector $w_\delta \in \mathbb{R}^n$ such that, setting

$$C_\delta := C(r_\delta) + w_\delta$$

and denoting by D_δ^1 and D_δ^2 the bases of C_δ and by L_δ its lateral boundary (see Figure 1), the following hold:

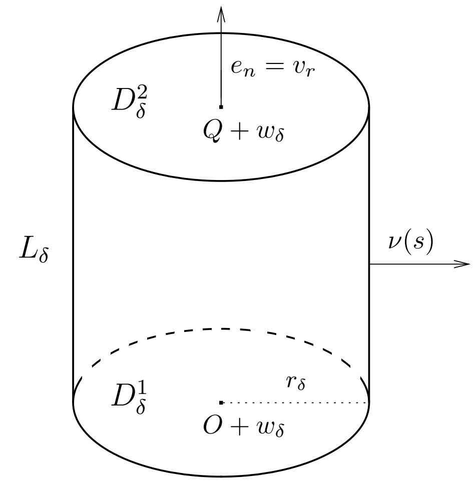

FIGURE 1. The cylinder C_δ

$$\begin{aligned} & C_\delta \subset \Omega\,, \\ & \mathcal{H}^{n-1}(\partial C_\delta \cap \mathbb{L}(B)) = \mathcal{H}^{n-1}(\partial C_\delta)\,, \\ & ⨍_{D_\delta^1} |\widetilde{B}(s) - A_1| d\mathcal{H}^{n-1}(s) + ⨍_{D_\delta^2} |\widetilde{B}(s) - A_2| d\mathcal{H}^{n-1}(s) < \delta\,. \end{aligned} \tag{3.11}$$

By Lemma 2.1 we get

$$\int_{\partial C_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s) = 0\,. \tag{3.12}$$

We write (3.12) as the sum of three contributions as follows

$$\begin{aligned} & \int_{\partial C_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s) = \int_{D_\delta^1} \widetilde{B}(s) \cdot (-e_n)\, d\mathcal{H}^{n-1}(s) + \\ & \int_{D_\delta^2} \widetilde{B}(s) \cdot e_n\, d\mathcal{H}^{n-1}(s) + \int_{L_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s)\,, \end{aligned} \tag{3.13}$$

where $\nu(s) = \frac{1}{\rho(s)}(s_1, \ldots, s_{n-1}, 0)$ on L_δ. On the other hand we have

$$\begin{aligned} \int_{D_\delta^1} \widetilde{B}(s) \cdot (-e_n)\, d\mathcal{H}^{n-1}(s) &= \mathcal{H}^{n-1}(D_\delta^1)\, A_1^\delta \cdot (-e_n)\,, \\ \int_{D_\delta^2} \widetilde{B}(s) \cdot e_n\, d\mathcal{H}^{n-1}(s) &= \mathcal{H}^{n-1}(D_\delta^1)\, A_2^\delta \cdot e_n\,, \end{aligned} \tag{3.14}$$

where A_1^δ, $A_2^\delta \in \mathbb{M}^{n\times n}$ and, by (3.11), are such that $|A_1^\delta - A_1| + |A_2^\delta - A_2| < \delta$. Then we set

$$u_\delta := \int_{L_\delta} \widetilde{B}(s) \cdot \nu(s)\, d\mathcal{H}^{n-1}(s)\,.$$

Dividing the right hand side in (3.13) by $\mathcal{H}^{n-1}(D^1_\delta)$ and using (3.12) and (3.14), we obtain

$$(A_2^\delta - A_1^\delta)e_n + \frac{u_\delta}{\mathcal{H}^{n-1}(D^1_\delta)} = 0\,. \tag{3.15}$$

Now recall that e_n is orthogonal to π_r, $\mathrm{rank}(A_2 - A_1) = n$ and $(A_2 - A_1)(\pi_r) = \pi_r$. It follows that $(A_2 - A_1)\cdot e_n \notin \pi_r$, and hence $(A_2^\delta - A_1^\delta)\cdot e_n \notin \pi_r$ for δ small enough. On the other hand we have that $u_\delta \in \pi_r$, the hyperplane π_r being preserved under the action of every $A_i \in K$. Then, for sufficiently small δ, (3.15) gives a contradiction. $\square$

Remark 3.2. The assumption of boundedness required for the set K, in Theorem 1.8, can be actually removed. The previous proof, indeed, can be adapted to L^1_{loc} matrix fields by suitable modifications.

Remark 3.3. Theorem 1.8, specialized to the case $m = n = 2$, gives a rigidity result which was already known in the setting of the "gradient problem" (see [8], Lemma 5).

4. Approximate solutions

In this section we give a refinement of Lemma 1.5 in [10] in which the authors give an explicit example of a set which is non-rigid for approximate solutions. Their construction is set in $\mathbb{M}^{3\times3}$, but it can be extended to the case $m, n \geq 3$ by slight modifications. It actually provides an algorithm (similar to that of Tartar's for the gradients, [17]), which allows us to find approximate solutions for a large class of sets K. Similar constructions can be found in the works of several authors, see [2], [7], [12], [13], [15]. The particular case here resembles the construction in [12]. The key point is the following lemma.

Lemma 4.1. *Let $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times3}$ be a set of pairwise rank-3 connected matrices. If there exist three matrices $S_1, S_2, S_3 \in \mathbb{M}^{3\times3}$ which satisfy the conditions*

$$\det(A_i - S_i) = 0, \quad \textit{for } i = 1,2,3\,, \tag{4.1}$$

$$S_i = q_{i-1}A_{i-1} + (1 - q_{i-1})S_{i-1} \qquad \mathrm{mod}\ 3\,, \quad \textit{for } i = 1,2,3\,, \tag{4.2}$$

for some $q_i \in (0,1)$, then the set K is non-rigid for approximate solutions.

The proof of this lemma relies on works on multiple scales. We refer to [3] and [6] for a general treatment and to [10] and [17] for the case of interest here. We simply remark that condition (4.1) is used to laminate A_i and S_i in some direction belonging to $\mathrm{Ker}(A_i - S_i)$, while (4.2) is used to construct a sequence which "approaches" the set K in the sense of

Lemma 1.5. This is the strategy used in [10] where the authors make an explicit choice of the matrices A_i and S_i. Theorem 1.9, given in the introduction, characterizes all possible triples $\{A_1, A_2, A_3\}$ which one can obtain in this way, and it is a corollary of the following proposition.

Proposition 4.2. *Let q_1, q_2, $q_3 \in (0,1)$ be given. Let $A_1 = 0$ and $A_2 = I$ in $\mathbb{M}^{3\times 3}$. Then there exist $S_1, S_2, S_3, A_3 \in \mathbb{M}^{3\times 3}$ satisfying conditions* (4.1) *and* (4.2) *of* Lemma 4.1, *if and only if A_3 is of the form*

$$A_3 = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} G - q_2(1-q_3)I\right], \tag{4.3}$$

where G is an arbitrary matrix in $GL(3)$ and the λ_i's are defined as follows

$$\lambda_1 = 0\,, \quad \lambda_2 = \frac{1}{1-q_1}\,, \quad \lambda_3 = \frac{q_2}{q_1 + q_2 - q_1 q_2}\,.$$

Proof. We rewrite (4.2) more explicitly:

$$\begin{aligned} S_2 &= q_1 A_1 + (1-q_1)S_1 = (1-q_1)S_1, \\ S_3 &= q_2 A_2 + (1-q_2)S_2 = q_2 I + (1-q_2)(1-q_1)S_1, \\ S_1 &= q_3 A_3 + (1-q_3)S_3 = q_3 A_3 + q_2(1-q_3)I + (1-q_1)(1-q_2)(1-q_3)S_1\,. \end{aligned} \tag{4.4}$$

Now let λ_i be the eigenvalues of S_1, then by (4.1) and (4.4) we get

$$\begin{aligned} &\det(A_1 - S_1) = 0 \Longleftrightarrow \det(-S_1) = 0 \Longleftrightarrow \lambda_1 = 0\,, \\ &\det(A_2 - S_2) = 0 \Longleftrightarrow \det(I - (1-q_1)S_1) = 0 \Longleftrightarrow \lambda_2 = \frac{1}{1-q_1}\,. \end{aligned}$$

Moreover one can check that

$$\det(A_3 - S_3) = 0 \Longleftrightarrow \det[(q_1 + q_2 - q_1 q_2)S_1 - q_2 I] = 0 \Longleftrightarrow \lambda_3 = \frac{q_2}{q_1 + q_2 - q_1 q_2}\,.$$

Note that the λ_i's are all distinct, since $q_1, q_2, q_3 \in (0,1)$. Therefore the matrix S_1 is diagonalizable. Hence for any $S_1 \in \{G^{-1}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)G,\ G \in GL(3)\}$, the matrices A_3, S_2, S_3 are uniquely determined by (4.4). In particular A_3 is of the form (4.3). Conversely, for any A_3 of the form (4.3), the matrices S_1, S_2, S_3 are uniquely determined and conditions (4.1) and (4.2) are satisfied. □

Proof of Theorem 1.9. Let $A, M, N \in \mathbb{M}^{3\times 3}$ satisfy the assumptions of Theorem 1.9. Without loss of generality we can assume that $M = 0$ and $N = I$. Notice that A is of the form (4.3). Then, by Proposition 4.2 and Lemma 4.1, the set $K = \{0, I, A\}$ is non-rigid for approximate solutions. □

Remark 4.3. If we choose $G = I$ in (4.3) the set K reduces itself to that given in [10], that is

$$A_3 = \mathrm{diag}\left(\frac{-q_2(1-q_3)}{q_3}, \frac{q_1+q_3-q_1q_3}{q_3(1-q_1)}, \frac{q_2}{q_1+q_2-q_1q_2}\right),$$
$$S_1 = \mathrm{diag}\left(0, \frac{1}{1-q_1}, \frac{q_2}{q_1+q_2-q_1q_2}\right),$$
$$S_2 = \mathrm{diag}\left(0, 1, \frac{q_2(1-q_2)}{q_1+q_2-q_1q_2}\right),$$
$$S_3 = \mathrm{diag}\left(q_2, 1, \frac{q_2}{q_1+q_2-q_1q_2}\right).$$

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